TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 5 2020

Scientific Calculator Assumed

Rates of Change and Differential Equations

STUDENT'S NAME Solutions

DATE: Monday 24 August **TIME:** 50 minutes **MARKS:** 47

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: **Three Scientific Calculators**, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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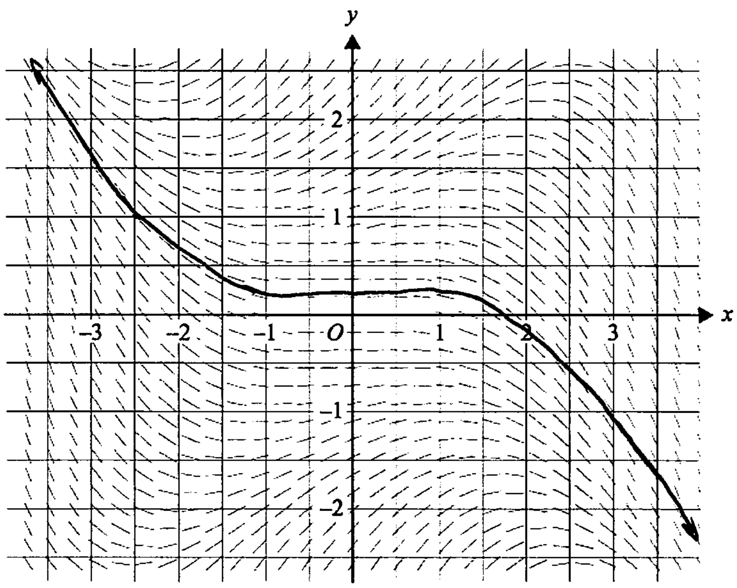
1. (3 marks)



The direction field for a certain differential equation is shown above.

(a) Sketch the solution curve to the differential equation that passes through the point $(-2.5, 1)$. [2]

(b) Which of the following points does it pass through? [1]

A. (0, 2) **B.** (1, 1) **C.** (3, −1) **D.** (3, −0.5) **E.** (−0.5, 2)

2. (5 marks)

For the differential equation $\frac{dy}{dx} = \frac{1+y^2}{2xy}$, solve for y in terms of x, given that when $x = 1, y = -1$.

$$\Rightarrow \int \frac{2y}{1+y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|1+y^2| = \ln|x| + c$$

At $(1,-1)$ we get

$$\ln 2 = c$$

$$\therefore \ln|1+y^2| = \ln|x| + \ln 2$$

$$\Rightarrow 1+y^2 = 2x$$

$$\Rightarrow y = \pm\sqrt{2x-1}$$

but $y = -1$ when $x = 1$

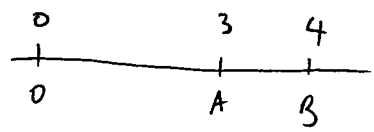
$$\therefore y = -\sqrt{2x-1}$$

3. (11 marks)

A small particle, P, describes simple harmonic motion along a straight with centre O. Two points, A and B, lie on this straight line with A between O and B such that $OA = 3\,m$ and $AB = 1\,m$. At A the speed of the particle is 32 ms^{-1} and at B its speed is 24 ms^{-1}.

(a) Using the equation $v^2 = k^2(A^2 - x^2)$, determine the value of A and k. [4]

$$\Rightarrow \quad 32^2 = k^2(A^2 - 3^2) \quad (1)$$
$$24^2 = k^2(A^2 - 4^2) \quad (2)$$



Now $\frac{(1)}{(2)} \Rightarrow \frac{16}{9} = \frac{A^2 - 9}{A^2 - 16}$

$$\Rightarrow \quad 16(A^2 - 16) = 9(A^2 - 9)$$
$$7A^2 = 175$$
$$A = \pm 5$$

So $24^2 = k^2(5^2 - 4^2)$

$k = \pm 8$

(b) Determine the period of the motion [2]

$$T = \frac{2\pi}{8} = \frac{\pi}{4} \text{ sec}$$

(c) Determine the maximum speed of P [2]

max when $x = 0 \Rightarrow v^2 = 8^2(5^2 - 0^2)$

$\therefore v_{max} = 40$ m/s

(d) Determine the time to travel from A to B [3]

$$x(t) = 5 \sin 8t$$

$$\Rightarrow \quad t = \frac{1}{8} \sin^{-1}\left(\frac{x}{5}\right)$$

So $t_4 - t_3 = \frac{1}{8} \sin^{-1}\left(\frac{4}{5}\right) - \frac{1}{8} \sin^{-1}\left(\frac{3}{8}\right)$

$= 0.23$ sec

4. (8 marks)

Audrey's activity is to ride a mini speedboat. To stop at the correct boat dock, she needs to stop the engine and allow the boat to be slowed down by air and water resistance. At time t seconds after the engine has been stopped, the acceleration of the boat, a ms^{-2}, is related to its velocity, v ms^{-1}, by

$$a = -\frac{1}{10}\sqrt{196 - v^2} \quad \text{(you may need the integral } \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\text{)}$$

Audrey stops the engine when the speedboat is travelling at 7 metres per second.

(a) Determine an equation for velocity in terms of time. [3]

$$\frac{dv}{dt} = -\frac{1}{10}\sqrt{196 - v^2}$$

$$\int \frac{dv}{\sqrt{196 - v^2}} = \int -\frac{1}{10}\, dt$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{v}{14}\right) = -\frac{t}{10} + c \quad , \quad \text{but } t = 0,\ v = 7 \Rightarrow c = \frac{\pi}{6}$$

$$\text{So} \quad \frac{v}{14} = \sin\left(\frac{-t}{10} + \frac{\pi}{6}\right)$$

$$v = 14 \sin\left(\frac{-t}{10} + \frac{\pi}{6}\right)$$

(b) Determine the time it takes for the speedboat to come to rest. Give your answer in seconds in terms of π. [2]

rest when $v = 0$

$$\Rightarrow \quad 0 = 14 \sin\left(\frac{-t}{10} + \frac{\pi}{6}\right)$$

$$\Rightarrow \quad 0 = \frac{-t}{10} + \frac{\pi}{6}$$

$$\Rightarrow \quad t = \frac{10\pi}{6}$$

$$= \frac{5\pi}{3} \quad \text{sec}$$

(c) Calculate the distance it takes the speedboat to come to rest, from when the engine is stopped. Give your answer in metres correct to one decimal place. [3]

$$\text{dist} = \int_0^{5\pi/3} 14 \sin\left(\frac{-t}{10} + \frac{\pi}{6}\right) dt$$

$$= 14 \left[10 \cos\left(\frac{-t}{10} + \frac{\pi}{6}\right)\right]_0^{5\pi/3}$$

$$= 140 \left[\cos\left(\frac{-5\pi}{30} + \frac{\pi}{6}\right) - \cos\frac{\pi}{6} \right]$$

$$= 140 \left[1 - \frac{\sqrt{3}}{2} \right]$$

$$\approx 18.8 \text{ m}$$

5. (10 marks)

The population of a culture is represented by the equation $N(t) = \dfrac{20}{1+10e^{-\frac{t}{100}}}$, where N is the number of individuals (in thousands) at any time t hours.

(a) When will the population reach 5000? [3]

$$\Rightarrow \quad 5 = \frac{20}{1+10e^{-t/100}}$$

$$\Rightarrow \quad 1+10e^{-t/100} = \frac{20}{5}$$

$$\Rightarrow \quad t = \ln\left(\frac{4-1}{10}\right) \times (-100) \qquad t = 120.40 \text{ hrs}$$

(b) Show that the rate of growth $\dfrac{dN}{dt} = kN(20-N)$ and determine the value of the constant k. All working must be shown in order to receive full marks. [4]

$$N = 20\left(1+10e^{-t/100}\right)^{-1}$$

$$\frac{dN}{dt} = -20\left(1+10e^{-t/100}\right)^{-2} \cdot \frac{-10}{100} e^{-t/100}$$

$$= \frac{-20}{-100}\left(\frac{20}{N}\right)^{-2} \cdot \left(\frac{20}{N} - 1\right)$$

$$= \frac{1}{5} \frac{N^2}{20^2} \left(\frac{20-N}{N}\right)$$

$$= \frac{1}{2000} N(20-N) \qquad \Rightarrow \quad k = \frac{1}{2000}$$

$$N = \frac{20}{1+10e^{-t/100}}$$

$$1+10e^{-t/100} = \frac{20}{N}$$

$$10e^{-t/100} = \frac{20}{N} - 1$$

(c) Given that the population after 5 hours is approximately 1903, calculate the approximate increase in the population during the following 5 hours using the incremental formula. Give your answer to the nearest integer. [3]

$$\frac{\delta N}{\delta t} \approx \frac{dN}{dt}$$

$$\delta N \approx \frac{1}{2000} 1.903(20 - 1.903)(5)$$

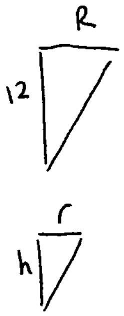
$$= 0.086076$$

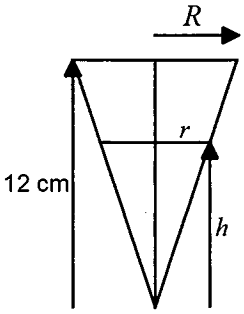
$$\approx 86$$

6. (10 marks)

The height of a conical glass is 12 cm with a base radius R cm. Water is being poured in at a constant rate of k cm^3 min^{-1} and the glass is filled in 2 minutes. (see diagram).

(a) Determine $\dfrac{dh}{dt}$ when the glass is filled to one-half of its height? Instant $(h = 6)$ [5]





$$\frac{r}{h} = \frac{R}{12}$$

$$r = \frac{Rh}{12}$$

$$\frac{dV}{dt} = k \text{ cm}^3/\text{min}$$

$\therefore$ Full vol $= 2k$

$$\Rightarrow 2k = \frac{\pi}{3} R^2 (12)$$

$$\frac{k}{2\pi} = R^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \frac{R^2 h^3}{144}$$

R is constant

$$\frac{dV}{dt} = \frac{\pi}{144} \cdot R^2 h^2 \frac{dh}{dt}$$

at instant

$$\cancel{k} = \frac{\cancel{\pi}}{144}\left(\frac{\cancel{k}}{2\cancel{\pi}}\right) \cdot (6)^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = 8 \text{ cm}/\text{min}$$

(b) Show that $\dfrac{dh}{dt} = 2^{\frac{5}{3}}$ cm min^{-1} when half of the water has been poured in? [5]

$\frac{1}{2}$ water $\Rightarrow V = k$

From part (a)

$$\frac{dV}{dt} = \frac{\cancel{\pi}}{144}\left(\frac{k}{2\cancel{\pi}}\right) h^2 \frac{dh}{dt}$$

Instant

$$\Rightarrow \cancel{k} = \frac{\cancel{k} h^2}{288} \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{288}{h^2}$$

$$= \frac{288}{864^{2/3}}$$

$$= \frac{2^5 \times \cancel{3^2}}{2^{10/3} \times \cancel{3^{6/3}}} = 2^{5/3}$$

$864 = 2^2 \times 6^3$

$= 2^5 \times 3^3$

$288 = 2^5 \times 3^2$

Also $V = \dfrac{\pi}{3} \dfrac{R^2 h^3}{144}$

as sim Δs' $\dfrac{r}{h} = \dfrac{R}{12}$

and $R^2 = \dfrac{k}{2\pi}$

$$\Rightarrow V = \frac{\cancel{\pi}}{3 \times 144}\left(\frac{k}{2\cancel{\pi}}\right) h^3$$

Now $\cancel{k} = \dfrac{\cancel{k}}{864} h^3$

$$\Rightarrow h = (864)^{1/3}$$